

The para-HK/QK correspondence

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January 20, 2016

Abstract

We generalise the hyper-Kähler/quaternionic Kähler (HK/QK) correspondence to include para-geometries, and present a new concise proof that the target manifold of the HK/QK correspondence is quaternionic Kähler. As an application, we construct one-parameter deformations of the temporal and Euclidean supergravity c -map metrics and show that they are para-quaternionic Kähler.

Introduction

In the original HK/QK correspondence developed by Haydys in [Ha] one starts with a hyper-Kähler manifold endowed with a *rotating*¹ Killing vector field and constructs a conical hyper-Kähler manifold of four dimensions higher, such that the original manifold can be recovered via the hyper-Kähler quotient construction². This conical hyper-Kähler manifold is then locally a Swann bundle over a quaternionic Kähler manifold which is again of four dimensions lower, i.e. of the same dimension as the original hyper-Kähler manifold. The construction of this quaternionic Kähler manifold from the original hyper-Kähler manifold is called the

¹This means that the vector field preserves one of the three complex structures while acting as an infinitesimal rotation on the other two.

²With non-zero choice of level set for the homogeneous hyper-Kähler moment map.

HK/QK correspondence. This correspondence was generalised in [ACM, ACDM] to include quaternionic Kähler manifolds of negative scalar curvature as well as pseudo-Riemannian hyper-Kähler and quaternionic Kähler manifolds, and has been formulated in terms of the associated twistor spaces in [Hi].

The main goal of this paper is to generalise the HK/QK correspondence to include para-versions of hyper-Kähler and quaternionic Kähler geometry. These are defined as follows³.

Definition 1 . A *para-hyper-Kähler* manifold (M, g, J_1, J_2, J_3) is a pseudo-Riemannian manifold (M, g) endowed with three skew-symmetric endomorphism fields $J_1, J_2, J_3 \in \Gamma(\text{End } TM)$ that satisfy the para-quaternion algebra

$$J_1^2 = \epsilon_1 Id_{TM}, \quad J_2^2 = \epsilon_2 Id_{TM}, \quad J_3^2 = \epsilon_3 Id_{TM}, \quad J_1 J_2 = J_3, \quad (1)$$

where $(\epsilon_1, \epsilon_2, \epsilon_3)$ is a permutation of $(-1, 1, 1)$, and such that the corresponding fundamental two-forms are closed.

Definition 2 . A *para-quaternionic Kähler* manifold (M, g, Q) of dimension $4n > 4$ is a pseudo-Riemannian manifold (M, g) with non-zero scalar curvature endowed with a parallel skew-symmetric rank-three subbundle $Q \subset \text{End } TM$ that is locally spanned by three endomorphism fields $J_1, J_2, J_3 \in \Gamma(Q)$ that satisfy the para-quaternion algebra (1).

The curvature tensor of a para-quaternionic Kähler manifold of dimension $4n > 4$ admits the decomposition [AC1]

$$R = \nu R_0 + W, \quad (2)$$

where $\nu := \text{scal}/(4n(n+2))$, R_0 is the curvature tensor of para-quaternionic projective space HP^n and W is a trace-free Q -invariant algebraic curvature tensor. In dimension four we define a para-quaternionic Kähler manifold to be a pseudo-Riemannian manifold satisfying Definition 2 and that admits the decomposition (2) of the curvature tensor.

In this paper we will construct what will be referred to as the *para-hyper-Kähler/para-quaternionic Kähler (para-HK/QK) correspondence*. This construction maps a para-hyper-Kähler manifold of dimension $4n$ to a para-quaternionic Kähler manifold of the same dimension. We will see that many of the arguments involved in the HK/QK correspondence

³An equivalent definition of a para-hyper-Kähler manifold of any dimension and a para-quaternionic Kähler manifold of dimension larger than 4 is that the holonomy group is contained in $Sp(2n, \mathbb{R}) \subset SO(2n, 2n)$ and $Sp(2n, \mathbb{R}) \cdot Sp(2, \mathbb{R}) \subset SO(2n, 2n)$ respectively [AC1].

can be directly applied to the para-HK/QK correspondence by flipping certain signs. We will therefore present a unified discussion of both correspondences which we will refer to jointly as the ε -HK/QK correspondence, where the parameter ε distinguishes between the two correspondences according to the rule

$$\varepsilon = \begin{cases} -1 & \text{HK/QK correspondence} \\ +1 & \text{Para-HK/QK correspondence} . \end{cases}$$

Similarly, we will use the terminology ε -hyper-Kähler and ε -quaternionic Kähler to refer to a hyper-Kähler or quaternionic Kähler manifold in the case $\varepsilon = -1$, and a para-hyper-Kähler or para-quaternionic Kähler manifold in the case $\varepsilon = 1$. We will formulate the ε -HK/QK correspondence in terms of a rank-one principal bundle P over the ε -hyper-Kähler base manifold M , rather than considering a conical ε -hyper-Kähler manifold. The ε -quaternionic Kähler target manifold M' is then a codimension one submanifold of P . This is summarised in the following diagram:

$$\begin{array}{ccc} & P_{4n+1} & \\ \swarrow & & \searrow \\ M_{4n} & \xrightarrow{\varepsilon\text{-HK/QK}} & M'_{4n} \end{array}$$

Using the rank-one principle bundle P over M , we prove that M' inherits an ε -quaternionic Kähler structure. This gives, in particular, a new concise proof of the original HK/QK correspondence. Compared to [Ha, ACM, ACDM] this proof is closer to that of [MS1, MS2] where the HK/QK correspondence is incorporated into Swann's twist formalism.

In the para-HK/QK correspondence the rotating Killing vector field on the para-hyper-Kähler base manifold preserves either a complex or a para-complex structure. In this sense the ε -HK/QK correspondence can be split into three distinct subcases:

- (i) The HK/QK correspondence induced by a holomorphic vector field.
- (ii) The para-HK/QK correspondence induced by a holomorphic vector field.
- (iii) The para-HK/QK correspondence induced by a para-holomorphic vector field.

We will show that in cases (i) and (ii) the ε -quaternionic Kähler target manifold admits an integrable complex structure, whilst in case (iii) the para-quaternionic Kähler target manifold

admits an integrable para-complex structure. In all cases the integrable structure is induced by the structure on the base manifold that is preserved by the rotating Killing vector field, and is compatible with the ε -quaternionic structure.

Para-quaternionic Kähler manifolds have recently appeared in the physics literature in the context of the *local temporal* (supergravity) *c-map* and the *local Euclidean* (supergravity) *c-map* [CDMV]. The local temporal *c-map* is a map from a projective special Kähler manifold of dimension $2n$ to a para-quaternionic Kähler manifold of dimension $4n+4$ that is induced by the dimensional reduction of 4D, $\mathcal{N} = 2$ Minkowskian local vector-multiplets over a timelike circle. Similarly, the local Euclidean *c-map* is a map from a projective special para-Kähler manifold to a para-quaternionic Kähler manifold (with the same dimensions as above) that is induced by the dimensional reduction of 4D, $\mathcal{N} = 2$ Euclidean local vector-multiplets over a spacelike circle. We will see that both maps can be understood geometrically in terms of the para-HK/QK correspondence, with the local temporal *c-map* corresponding to case (ii) and the local Euclidean *c-map* corresponding to case (iii). This provides an alternative proof that the target manifolds in both cases are para-quaternionic Kähler. Moreover, we will use the para-HK/QK correspondence to construct one-parameter deformations of the local temporal and Euclidean *c-map* metrics, which are para-quaternionic Kähler by construction. This is analogous to the proof given in [ACDM] that the one-loop deformed local spatial *c-map* metric, which first appeared in the physics literature in [RSV], is quaternionic Kähler⁴. To our knowledge the deformations of the local temporal and Euclidean *c-map* metrics that we present here have not previously appeared in the literature.

Acknowledgements

We would like to thank Vicente Cortés for suggesting the topic of this paper and for useful discussions. The work of M.D. was supported by the RTG 1670 “Mathematics inspired by String Theory,” funded by the Deutsche Forschungsgemeinschaft (DFG). The work of O.V. was supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 “Particles, Strings and the Early Universe.”

⁴The fact that the rigid *c-map* metric can be obtained from the one-loop deformed local spatial *c-map* metric via the QK/HK correspondence was previously shown in [APP].

1 The ε -HK/QK correspondence

Let (M, g, J_1, J_2, J_3) be an ε -hyper-Kähler manifold with ε -hyper-complex structure satisfying (1) with $(\epsilon_1, \epsilon_2, \epsilon_3)$ a permutation of $(-1, \varepsilon, \varepsilon)$. We will use the following convention for the definition of the ϵ_α -Kähler forms⁵:

$$\omega_\alpha := -\epsilon_\alpha g(J_\alpha \cdot, \cdot) , \quad \alpha = 1, 2, 3 . \quad (3)$$

Notice that

$$\epsilon_3 = -\epsilon_1 \epsilon_2 . \quad (4)$$

This means that J_1, J_2, J_3 are complex or para-complex according to the rule

	(ϵ_1, ϵ_2)	complex	para-complex
(i)	$(-1, -1)$	J_1, J_2, J_3	
(ii)	$(-1, +1)$	J_1	J_2, J_3
(iii)	$(+1, -1)$	J_2	J_1, J_3
(iv)	$(+1, +1)$	J_3	J_1, J_2

and fulfil

$$J_\alpha J_\beta = -J_\beta J_\alpha = \epsilon_3 \epsilon_\gamma J_\gamma \quad (5)$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. The ordering of the endomorphisms in the above table will be important later. In particular, the ϵ_1 -complex structure J_1 induces an integrable ϵ_1 -complex structure on the ε -quaternionic Kähler target manifold. This means that for the para-HK/QK correspondence there are two possibilities: either the para-quaternionic Kähler target manifold admits an integrable complex structure, which corresponds to case (ii), or an integrable para-complex structure, which corresponds to cases (iii) and (iv). In the discussion that follows there will be no distinction between cases (iii) and (iv), which are equivalent up to relabelling of J_2 and J_3 . The three distinct cases (i), (ii) and (iii) correspond to the three subcases of the ε -HK/QK correspondence described in the Introduction.

In order to perform the ε -HK/QK correspondence there must exist a real-valued function $f \in C^\infty(M)$ such that the vector field

$$Z := -\omega_1^{-1}(df) \quad (\text{i.e. } \omega_1(Z, \cdot) = -df) \quad (6)$$

⁵Note that this convention differs from the convention in [CDMV] by a minus sign.

is timelike or spacelike, Killing, J_1 -holomorphic (that is, $\mathcal{L}_Z J_1 = 0$), and satisfies

$$\mathcal{L}_Z J_2 = \epsilon_1 2J_3 . \quad (7)$$

Such a vector field is called a *rotating* vector field. We define

$$f_1 := f - \frac{g(Z, Z)}{2} , \quad \beta := g(Z, \cdot) , \quad \sigma := \text{sign} f , \quad \sigma_1 := \text{sign} f_1 , \quad (8)$$

and assume that σ and σ_1 are constant and non-vanishing.

Up to a minus sign, the function f is an ϵ_1 -Kähler moment map of Z with respect to ω_1 . Note the simple but important fact that such a moment map is only defined up to a shift by a real constant. This will lead to a one-parameter deformation of the resulting ϵ -quaternionic Kähler metric.

Let $\pi : P \rightarrow M$ be a rank-one principal bundle over M , and let $\eta \in \Omega^1(P)$ be a principal connection on P with curvature

$$d\eta = \pi^* \left(\omega_1 - \frac{1}{2} d\beta \right) . \quad (9)$$

Let $\tilde{Y} \in \Gamma(\ker \eta)$ denote the horizontal lift to P of any vector field Y on M , and let X_P denote the fundamental vector field of the principal action on P normalised such that $\eta(X_P) = 1$. On P we define the metric

$$g_P := \frac{2}{f_1} \eta^2 + \pi^* g , \quad (10)$$

the vector field

$$Z_1^P := \tilde{Z} + f_1 X_P , \quad (11)$$

and one-forms

$$\begin{aligned} \theta_0^P &:= \frac{1}{2} df , \\ \theta_1^P &:= \eta + \frac{1}{2} \beta , \\ \theta_2^P &:= -\frac{\epsilon_2}{2} \pi^* \omega_3(Z, \cdot) , \\ \theta_3^P &:= \frac{\epsilon_2}{2} \pi^* \omega_2(Z, \cdot) . \end{aligned} \quad (12)$$

Here we do not explicitly write the pull-back symbol in front of the functions f, f_1, β .

In the following, we will prove in particular that

$$\begin{aligned} g' &:= \frac{1}{2|f|} \left(g_P - \frac{2}{f} ((\theta_1^P)^2 - \epsilon_1(\theta_0^P)^2 - \epsilon_2(\theta_3^P)^2 - \epsilon_3(\theta_2^P)^2) \right) \Big|_{M'} \\ &= \frac{1}{2|f|} \left(g_P - \frac{2\epsilon_1}{f} \sum_{a=0}^3 \epsilon_a (\theta_a^P)^2 \right) \Big|_{M'} \quad (\epsilon_0 := -1) \end{aligned} \quad (13)$$

defines an ε -quaternionic Kähler metric on any codimension one submanifold $M' \subset P$ that is transversal to Z_1^P .

For the proof of the above statement, we will now gather some relevant geometric properties of the original ε -hyper-Kähler manifold M that are implied by the existence of the rotating Killing vector field Z .

Using Z one may define a rank-four ‘vertical’ distribution and its ‘horizontal’ orthogonal complement in TM :

$$\mathcal{D}^v := \text{span}\{Z, J_1Z, J_2Z, J_3Z\}, \quad \mathcal{D}^h := (\mathcal{D}^v)^{\perp_g} \subset TM,$$

in which case the tangent bundle decomposes as

$$TM = \mathcal{D}^v \oplus^{\perp_g} \mathcal{D}^h.$$

With respect to the frame (Z, J_1Z, J_2Z, J_3Z) on \mathcal{D}^v , the endomorphisms J_1, J_2, J_3 are represented respectively by the matrices

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -\epsilon_1\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & -\epsilon_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Let us define the following one-forms on M :

$$\begin{aligned} \theta_0 &:= \frac{1}{2}df = -\frac{1}{2}\omega_1(Z, \cdot) = \frac{\epsilon_1}{2}g(J_1Z, \cdot) &= \frac{\epsilon_1}{2}g(W, \cdot), \\ \theta_1 &:= \frac{1}{2}\beta = \frac{1}{2}g(Z, \cdot) &= \frac{\epsilon_1}{2}g(J_1W, \cdot), \\ \theta_2 &:= -\frac{\epsilon_2}{2}\omega_3(Z, \cdot) = -\frac{\epsilon_1}{2}g(J_3Z, \cdot) &= \frac{\epsilon_1}{2}g(J_2W, \cdot), \\ \theta_3 &:= \frac{\epsilon_2}{2}\omega_2(Z, \cdot) = -\frac{1}{2}g(J_2Z, \cdot) &= \frac{\epsilon_1}{2}g(J_3W, \cdot), \end{aligned}$$

where $W := J_1 Z$. Notice that the one-forms $(\theta_a^P)_{a=0,\dots,3}$ defined in Eq. (12) are precisely the pull-backs of the one-forms (θ_a) defined above with the exception of θ_1^P which has an additional η inserted.

Proposition 1. *The ε -hyper-Kähler metric can be written as*

$$g = \frac{4}{\beta(Z)} ((\theta_1)^2 - \epsilon_1(\theta_0)^2 - \epsilon_2(\theta_3)^2 - \epsilon_3(\theta_2)^2) + \check{g} , \quad (15)$$

where \check{g} is a symmetric rank-two tensor field that is invariant under Z and has four-dimensional kernel \mathcal{D}^v . The ε -Kähler forms are given by

$$\omega_\alpha := -\epsilon_\alpha g(J_\alpha \cdot, \cdot) = \frac{4}{\beta(Z)} (\epsilon_1 \epsilon_\alpha \theta_0 \wedge \theta_\alpha - \epsilon_2 \theta_\beta \wedge \theta_\gamma) + \check{\omega}_\alpha , \quad (16)$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$, and we have defined $\check{\omega}_\alpha := -\epsilon_\alpha \check{g}(J_\alpha \cdot, \cdot)$. The two-forms $(\check{\omega}_\alpha)_{\alpha=1,2,3}$ are degenerate but not invariant under Z .

Proof. Since g is ε -hyper-Hermitian $Z, J_1 Z, J_2 Z, J_3 Z$ are pairwise orthogonal. They have squared norm $g(Z, Z) = \beta(Z), -\epsilon_1 \beta(Z), -\epsilon_2 \beta(Z), -\epsilon_3 \beta(Z)$ respectively. The tensor field

$$\begin{aligned} \check{g} &= g - \frac{4}{\beta(Z)} ((\theta_1)^2 - \epsilon_1(\theta_0)^2 - \epsilon_2(\theta_3)^2 - \epsilon_3(\theta_2)^2) \\ &= g - \frac{1}{g(Z, Z)} ((Z^\flat)^2 - \epsilon_1(J_1 Z^\flat)^2 - \epsilon_2(J_2 Z^\flat)^2 - \epsilon_3(J_3 Z^\flat)^2) , \end{aligned}$$

where $Z^\flat = g(Z, \cdot)$ and $J_\alpha Z^\flat = g(J_\alpha Z, \cdot)$, has kernel \mathcal{D}^v . Since Z is Killing we have $\mathcal{L}_Z \beta = 0 = \mathcal{L}_Z \theta_1$, and since Z is J_1 -holomorphic we have $\mathcal{L}_Z \theta_0 = 0$. In addition we have $\mathcal{L}_Z J_2 = \epsilon_1 2J_3$ and $\mathcal{L}_Z J_3 = 2J_2$ and therefore

$$\mathcal{L}_Z \theta_2 = \epsilon_1 2\theta_3 , \quad \mathcal{L}_Z \theta_3 = 2\theta_2 .$$

From these expressions it easily follows that $\mathcal{L}_Z \check{g} = 0$.

Next, we calculate

$$J_\alpha^* \theta_0 = -\theta_\alpha , \quad J_\alpha^* \theta_\beta = -\epsilon_3 \epsilon_\gamma \theta_\gamma , \quad J_\alpha^* \theta_\alpha = -\epsilon_\alpha \theta_0 , \quad J_\alpha^* \theta_\gamma = \epsilon_3 \epsilon_\beta \theta_\beta$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. Using the fact that $\epsilon_\alpha \epsilon_\beta = -\epsilon_\gamma$ the expressions for the ε -Kähler forms are given by

$$\omega_\alpha = -\frac{4\epsilon_1 \epsilon_\alpha}{\beta(Z)} (-\theta_0 \wedge \theta_\alpha - \epsilon_3 \epsilon_\alpha \theta_\beta \wedge \theta_\gamma) + \check{\omega}_\alpha ,$$

which can be written as (16). □

Let us now turn our attention to the rank-one principal bundle $\pi : P \rightarrow M$.

Lemma 1. *The exterior derivatives of the one-forms $(\theta_\alpha^P)_{\alpha=1,2,3}$ defined on P in Eq. (12) are given by*

$$d\theta_\alpha^P = \epsilon_1 \epsilon_\alpha \pi^* \omega_\alpha, \quad \alpha = 1, 2, 3.$$

Proof. From the curvature of η we see immediately that

$$d\theta_1^P = d\eta + \frac{1}{2}d\beta = \pi^* \omega_1 = \epsilon_1 \epsilon_1 \pi^* \omega_1.$$

Next, note that

$$\mathcal{L}_Z \omega_3 = -\epsilon_1 2\omega_2, \quad \mathcal{L}_Z \omega_2 = -2\omega_3,$$

from which it follows that

$$\begin{aligned} d\theta_2^P &= -\frac{\epsilon_2}{2} d(\iota_Z \omega_3) = -\frac{\epsilon_2}{2} \mathcal{L}_Z \omega_3 = \epsilon_1 \epsilon_2 \omega_2, \\ d\theta_3^P &= \frac{\epsilon_2}{2} d(\iota_Z \omega_2) = \frac{\epsilon_2}{2} \mathcal{L}_Z \omega_2 = -\epsilon_2 \omega_3 = \epsilon_1 \epsilon_3 \omega_3. \end{aligned}$$

□

Let M' be a codimension one submanifold of P transversal to Z_1^P , that is $TP|_{M'} = TM' \oplus \mathbb{R}Z_1^P$. Let $pr_{TM'}$ denote the projection onto the first factor, i.e. the projection from TP to TM' along Z_1^P . On M' we define

$$X := pr_{TM'} \circ X_P|_{M'},$$

and for any vector field Y on M we define the vector field on M'

$$Y' := pr_{TM'} \circ \tilde{Y}|_{TM'}.$$

Since $pr_{TM'} Z_1^P = 0$ we have

$$X = pr_{TM'} \circ X_P|_{M'} = -\frac{1}{f_1} pr_{TM'} \circ \tilde{Z}|_{M'} = -\frac{1}{f_1} Z'. \quad (17)$$

We define the distributions

$$\mathcal{D}'^h := \text{span}\{Y' \mid Y \in \Gamma(\mathcal{D}^h)\} \subset TM',$$

and

$$\mathcal{D}'^v := \text{span}\{X, (J_1 Z)', (J_2 Z)', (J_3 Z)'\} = \text{span}\{Z', (J_1 Z)', (J_2 Z)', (J_3 Z)'\} \subset TM'.$$

We can then decompose the tangent space of M' as

$$TM' = \mathcal{D}'^v \oplus \mathcal{D}'^h.$$

Proposition 2. *An almost ε -hyper-complex structure (J'_1, J'_2, J'_3) on M' such that $J'^2_\alpha = \epsilon_\alpha Id_{TM'}$ and $J_1 J_2 = J_3$ is uniquely defined by*

$$J'_\alpha X = -\frac{1}{f_1} (J_\alpha Z)', \quad J'_\alpha (J_\beta Z)' = (J_\gamma Z)',$$

and

$$J'_\alpha(Y') = (J_\alpha Y)' \quad \text{for all } Y' \in \Gamma(\mathcal{D}'^h).$$

Proof. Since J_α preserves \mathcal{D}^h it follows that J'_α preserves \mathcal{D}'^h . It is also clear that J'_α preserves \mathcal{D}'^v . The matrices representing $J'_1|_{\mathcal{D}'^v}, J'_2|_{\mathcal{D}'^v}, J'_3|_{\mathcal{D}'^v}$ are given by Eq. (14), and for all $Y' \in \mathcal{D}'^h$ we have

$$J'_{\alpha_1} J'_{\alpha_2} Y' = J'_{\alpha_1} (J_{\alpha_2} Y)' = (J_{\alpha_1} J_{\alpha_2} Y)'.$$

Therefore (J'_1, J'_2, J'_3) fulfil the ε -quaternionic algebra

$$J'^2_1 = \epsilon_1 Id_{TM'}, \quad J'^2_2 = \epsilon_2 Id_{TM'}, \quad J'^2_3 = \epsilon_3 Id_{TM'}, \quad J'_1 J'_2 = J'_3.$$

□

Theorem 1. *Let (M, g, J_1, J_2, J_3) be an ε -hyper-Kähler manifold and let $f \in C^\infty(M)$ be a function on M that fulfils the assumptions stated above. Choose a rank-one principal bundle P with connection η and a submanifold $M' \subset P$ as above. Define*

$$Q := \text{span}\{J'_1, J'_2, J'_3\},$$

with J'_1, J'_2, J'_3 defined as above. Then (M', g', Q) with

$$g' := \frac{1}{2|f|} \left(g_P - \frac{2}{f} ((\theta_1^P)^2 - \epsilon_1 (\theta_0^P)^2 - \epsilon_2 (\theta_3^P)^2 - \epsilon_3 (\theta_2^P)^2) \right) \Big|_{M'} \quad (18)$$

is an ε -quaternionic Kähler manifold and X is Killing with respect to g' .

Proof for $\dim_{\mathbb{R}} M > 4$:

Substituting in (10) and then (15) we find

$$\begin{aligned} g' &= \frac{1}{2|f|} \left(\frac{2}{f_1} \eta^2 + \pi^* g - \frac{2}{f} ((\theta_1^P)^2 - \epsilon_1(\theta_0^P)^2 - \epsilon_2(\theta_3^P)^2 - \epsilon_3(\theta_2^P)^2) \right) \Big|_{M'} \\ &= \frac{1}{2|f|} \left(\frac{2}{f_1} \eta^2 + \frac{4}{\beta(Z)} ((\theta_1^P - \eta)^2 - \epsilon_1(\theta_0^P)^2 - \epsilon_2(\theta_3^P)^2 - \epsilon_3(\theta_2^P)^2) + \pi^* \check{g} \right. \\ &\quad \left. - \frac{2}{f} ((\theta_1^P)^2 - \epsilon_1(\theta_0^P)^2 - \epsilon_2(\theta_3^P)^2 - \epsilon_3(\theta_2^P)^2) \right) \Big|_{M'}. \end{aligned}$$

We then use the fact that

$$\frac{4}{\beta(Z)} - \frac{2}{f} = \frac{4f_1}{f\beta(Z)}, \quad \frac{4}{\beta(Z)} + \frac{2}{f_1} = \frac{4f}{f_1\beta(Z)},$$

and

$$\frac{2}{f_1} \eta^2 + \frac{4}{\beta(Z)} (\theta_1^P - \eta)^2 - \frac{2}{f} (\theta_1^P)^2 = \frac{4f_1}{f\beta(Z)} (\theta_1^P - \frac{f}{f_1} \eta)^2,$$

to write this as

$$\begin{aligned} g' &= \frac{1}{2|f|} \left(\frac{4f_1}{f\beta(z)} \left((\theta_1^P - \frac{f}{f_1} \eta)^2 - \epsilon_1(\theta_0^P)^2 - \epsilon_2(\theta_3^P)^2 - \epsilon_3(\theta_2^P)^2 \right) + \pi^* \check{g} \right) \Big|_{M'} \\ &= \lambda \sigma \sigma_1 ((\theta'_1)^2 - \epsilon_1(\theta'_0)^2 - \epsilon_2(\theta'_3)^2 - \epsilon_3(\theta'_2)^2) + \frac{1}{2|f|} \pi^* \check{g} \Big|_{M'}, \end{aligned}$$

where

$$\begin{aligned} \theta'_0 &:= \frac{1}{|f|} \sqrt{\left| \frac{2f_1}{\beta(Z)} \right|} \theta_0^P \Big|_{M'}, & \theta'_1 &:= \frac{1}{|f|} \sqrt{\left| \frac{2f_1}{\beta(Z)} \right|} \left(\theta_1^P - \frac{f}{f_1} \eta \right) \Big|_{M'}, \\ \theta'_2 &:= \frac{1}{|f|} \sqrt{\left| \frac{2f_1}{\beta(Z)} \right|} \theta_2^P \Big|_{M'}, & \theta'_3 &:= \frac{1}{|f|} \sqrt{\left| \frac{2f_1}{\beta(Z)} \right|} \theta_3^P \Big|_{M'}. \end{aligned}$$

and

$$\lambda := \text{sign} \beta(Z), \quad \sigma = \text{sign} f, \quad \sigma_1 = \text{sign} f_1.$$

Since Z_1^P lies in the kernel of $\theta_0^P, \theta_1^P - f/f_1 \eta, \theta_2^P, \theta_3^P$ the above splitting of g' corresponds to the splitting $TM' = \mathcal{D}'^v \oplus \mathcal{D}'^h$ given previously. Therefore the first summand is non-degenerate on \mathcal{D}'^v and has kernel \mathcal{D}'^h whilst the second summand is non-degenerate on \mathcal{D}'^h and has kernel \mathcal{D}'^v . Note that this already implies that g' is non-degenerate. In addition g' is invariant under X .

We will show that

$$\omega'_\alpha := -\epsilon_\alpha g'(J_\alpha \cdot, \cdot) = \frac{\sigma}{2} \epsilon_1 \epsilon_\alpha d\bar{\theta}_\alpha + \sigma \epsilon_2 \bar{\theta}_\beta \wedge \bar{\theta}_\gamma \quad (19)$$

where we have defined the one-forms

$$\bar{\theta}_\alpha := \frac{1}{f} \theta_\alpha^P \Big|_{M'} . \quad (20)$$

Differentiating gives

$$d\omega'_\alpha = 2\epsilon_3 (\epsilon_\gamma \bar{\theta}_\beta \wedge \omega'_\gamma - \epsilon_\beta \bar{\theta}_\gamma \wedge \omega'_\beta) . \quad (21)$$

From these expressions it follows immediately that the fundamental four-form

$$\Omega_4 := \sum_{\alpha=1,2,3} \epsilon_\alpha \omega'_\alpha \wedge \omega'_\alpha$$

is closed, and that the algebraic ideal generated by $(\omega'_1, \omega'_2, \omega'_3)$ in $\Omega^*(M')$ is a differential ideal. In dimensions greater than eight the closure of the fundamental four-form is enough to show that the metric is para-quaternionic Kähler. In dimension eight the closure of the fundamental four-form along with the fact that the fundamental two-forms generate a differential ideal is enough to show that the metric is ε -quaternionic Kähler. Both statements were originally stated in [S] for almost quaternionic pseudo-Hermitian manifolds. The proof in the appendix of [S] is based on complex representation theory that does not depend on the respective real form. Hence both statements can be generalised to the para-quaternionic Kähler case (see [DJS] for the first statement in the para-quaternionic case). This then completes the proof of Theorem 1.

It is left to verify equation (19). Since (J'_1, J'_2, J'_3) agrees with (J_1, J_2, J_3) on \mathcal{D}'^h we have

$$\pi^* (-\epsilon_\alpha \check{g})|_{M'} (J'_\alpha \cdot, \cdot) = \pi^* (-\epsilon_\alpha \check{g}(J_\alpha \cdot, \cdot))|_{M'} = \pi^* \check{\omega}_\alpha|_{M'} .$$

On \mathcal{D}'^v the vectors $X, J_1 X, J_2 X, J_3 X$ are pairwise orthogonal with respect to

$$(\theta'_1)^2 - \epsilon_1 (\theta'_0)^2 - \epsilon_2 (\theta'_3)^2 - \epsilon_3 (\theta'_2)^2 ,$$

and fulfil

$$\theta'_0(J'_1 X) = -\theta'_1(X) = \epsilon_2 \theta'_2(J'_3 X) = -\epsilon_2 \theta'_3(J'_2 X) = \frac{\lambda \sigma_1}{|f|} \sqrt{\left| \frac{\beta(Z)}{2f_1} \right|} \neq 0 .$$

We therefore have

$$J'_\alpha * \theta'_0 = -\theta'_\alpha, \quad J'_1 * \theta'_2 = -\theta'_3, \quad J'_2 * \theta'_3 = \epsilon_2 \theta'_1, \quad J'_3 * \theta'_1 = \epsilon_1 \theta'_2.$$

Next, we note that $\theta_\alpha^P = \pi^* \theta_\alpha$ for $\alpha = 0, 2, 3$ and

$$\begin{aligned} \left(\frac{2}{|f|\beta(Z)} \pi^* \theta_1 - \frac{\sigma}{f^2} \theta_1^P \right) \Big|_{M'} &= \left(\frac{2}{|f|\beta(Z)} (\theta_1^P - \eta) - \frac{\sigma}{f^2} \theta_1^P \right) \\ &= \left(\frac{1}{|f|} \frac{2f_1}{f\beta(Z)} (\theta_1^P - \frac{f}{f_1} \eta) \right) = \lambda \sigma \sigma_1 \frac{1}{|f|} \sqrt{\left| \frac{2f_1}{\beta(Z)} \right|} \theta'_1. \end{aligned}$$

From Lemma 1 we have

$$d\bar{\theta}_\alpha = \frac{1}{f} \epsilon_1 \epsilon_\alpha \pi^* \omega_\alpha - \frac{1}{f^2} df \wedge \theta_\alpha^P.$$

Putting everything together we find

$$\begin{aligned} \omega'_\alpha &= \lambda \sigma \sigma_1 (\epsilon_1 \epsilon_\alpha \theta'_0 \wedge \theta'_\alpha - \epsilon_2 \theta'_\beta \wedge \theta'_\gamma) + \frac{1}{2|f|} \pi^* \tilde{\omega}_\alpha \Big|_{M'} \\ &= \left(\frac{1}{2|f|} \pi^* \tilde{\omega}_\alpha + \frac{2}{|f|\beta(Z)} \pi^* (\epsilon_1 \epsilon_\alpha \theta_0 \wedge \theta_\alpha - \epsilon_2 \theta_\beta \wedge \theta_\gamma) - \frac{\sigma}{f^2} (\epsilon_1 \epsilon_\alpha \theta_0^P \wedge \theta_\alpha^P - \epsilon_2 \theta_\beta^P \wedge \theta_\gamma^P) \right) \Big|_{M'} \\ &= \left(\frac{1}{2|f|} \pi^* \omega_\alpha - \epsilon_1 \epsilon_\alpha \frac{\sigma}{2f^2} df \wedge \theta_\alpha^P + \epsilon_2 \frac{\sigma}{f^2} \theta_\beta^P \wedge \theta_\gamma^P \right) \Big|_{M'} \\ &= \frac{\sigma}{2} (\epsilon_1 \epsilon_\alpha d\bar{\theta}_\alpha + \epsilon_2 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma). \end{aligned}$$

This verifies Eq. (19) and ends the proof in dimension greater than four. \square

Proof for $\dim_{\mathbb{R}} M = 4$:

The proof in dimension four relies on the fact that on any submanifold M' of an ε -quaternionic Kähler manifold $(\tilde{M}, \tilde{g}, \tilde{Q})$ such that TM' is \tilde{Q} -invariant,

$$(Q' := \tilde{Q}|_{M'}, g' := \tilde{g}|_{M'})$$

defines an ε -quaternionic Kähler structure (see [M, Prop. 8] and references therein). This idea is taken from [MS2, Cor. 4.2.].

Assume that $\dim_{\mathbb{R}} M = 4$. Let $M_0 := \mathbb{R}^4$ be endowed with standard real coordinates (x, y, u, v) . Using $z := x + i_{\epsilon_1} y$ and $w := u - i_{\epsilon_1} v$, we define an ε -hyper-Kähler structure $(g_0, J_1^0, J_2^0, J_3^0)$ by

$$g_0 := dz d\bar{z} - \epsilon_2 dw d\bar{w}, \quad \omega_+^0 := \omega_2^0 + i_{\epsilon_1} \omega_3^0 = dz \wedge dw.$$

Let $f^0 := \epsilon_1 \epsilon_2 w \bar{w} \in C^\infty(M_0)$. This defines a J_1^0 -holomorphic vector field

$$Z^0 := -(\omega_1^0)^{-1}(df) = -2\epsilon_1 i_{\epsilon_1}(w \partial_w - \bar{w} \partial_{\bar{w}})$$

that fulfils $\mathcal{L}_Z \omega_+^0 = -2\epsilon_1 i_{\epsilon_1} \omega_+^0$ and, hence, $\mathcal{L}_{Z^0} J_2^0 = 2\epsilon_1 J_3^0$. The one-form $\eta_0^{M_0} := \frac{1}{2} \text{Im}(\bar{z} dz + \epsilon_2 \bar{w} dw)$ fulfils $d\eta_0^{M_0} = \omega_1^0 - \frac{1}{2} d(\iota_{Z^0} g_0)$ and we have $f_1^0 := f^0 - \frac{1}{2} g_0(Z^0, Z^0) = -\epsilon_1 \epsilon_2 w \bar{w}$.

Consider $(\tilde{M} := M \times \mathbb{R}^4, \tilde{g} := g + g_0, \tilde{f} := f + f^0)$ together with the ε -hyper-complex structure $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$. Let $\tilde{U} \subset \tilde{M}$ be a neighbourhood of $M = M \times \{0\} \subset \tilde{M}$ such that the signs of $\tilde{f}, \tilde{f}_1 := f_1 + f_1^0$ and $\tilde{f} - \tilde{f}_1$ restricted to \tilde{U} are constant. Then the restriction of the above data from \tilde{M} to \tilde{U} fulfils the assumptions of the ε -HK/QK correspondence. The restriction of $P \times \mathbb{R}^4$ defines a rank-one principal bundle \tilde{P} over \tilde{U} with connection $\tilde{\eta} = (\eta + \eta_0^{M_0})|_{\tilde{P}}$. The ε -HK/QK correspondence with the choices $(\tilde{P}, \tilde{\eta}, \tilde{M}' := M' \times \mathbb{R}^4)$ then defines an ε -quaternionic Kähler structure (\tilde{g}', \tilde{Q}) on the eight-dimensional manifold \tilde{M}' . The submanifold $M' = M' \times \{0\} \subset \tilde{M}'$ has a \tilde{Q} -invariant tangent bundle and, hence, $(M', \tilde{g}'|_{M'}, \tilde{Q}|_{M'})$ is ε -quaternionic Kähler. The one-forms $\tilde{\theta}_\alpha$ on \tilde{M}' obtained from the ε -HK/QK-correspondence (see Eq. (20)) restrict to the corresponding one-forms $\bar{\theta}_\alpha$ on M' . The latter one-forms define the ε -quaternionic Kähler structure (g', Q) on M' obtained from the ε -HK/QK correspondence by Eq. (19), which in particular shows that $(\tilde{g}'|_{M'}, \tilde{Q}|_{M'}) = (g', Q)$. \square

Remark 1. On any ε -quaternionic Kähler manifold one can define three one-forms $(\bar{\theta}_\alpha)_{\alpha=1,2,3}$ by

$$\nabla \cdot J'_\alpha = 2\epsilon_3 \epsilon_\alpha (\bar{\theta}_\beta(\cdot) J'_\gamma - \bar{\theta}_\gamma(\cdot) J'_\beta) , \quad (22)$$

which is equivalent to Eq. (21). Then the fundamental two-forms fulfil [AC2, Prop. 5]⁶

$$\frac{\nu}{2} \omega'_\alpha = -\epsilon_\alpha d\bar{\theta}_\alpha + 2\epsilon_3 \bar{\theta}_\beta \wedge \bar{\theta}_\gamma , \quad (23)$$

where $\nu := \frac{\text{scal}}{4n(n+2)}$ ($\dim_{\mathbb{R}} M' = 4n$) is the reduced scalar curvature. Comparing Eqs. (19) and (23) shows that the reduced scalar curvature of g' is $\nu = -\epsilon_1 4\sigma$.

Theorem 2. *Let (M', g', Q) be an ε -quaternionic Kähler manifold in the image of the ε -HK/QK correspondence as described above. The globally defined almost ϵ_1 -complex structure $J'_1 \in \Gamma(Q)$ is integrable.*

⁶Compared to [AC2] we have $\omega'_\alpha = \rho'_\alpha$ ^[AC2] and $\bar{\theta}_\alpha = -\frac{\epsilon_3 \epsilon_\alpha}{2} \omega_\alpha$ ^[AC2].

Proof. Let $a \in C^\infty(P)$ such that $X = (X_P - aZ_1P)|_{M'} \in \mathfrak{X}(M')$. We will identify the ε -quaternionic Kähler moment map associated with X and use it to prove that J'_1 is integrable.

The Killing vector field X satisfies

$$(\iota_X \bar{\theta})_{\alpha=1,2,3} = ((f'^{-1} - a'), 0, 0) \ , \quad (\iota_X d\bar{\theta})_{\alpha=1,2,3} = (f'^{-2} df', -\epsilon_2 a' 2\bar{\theta}_3, \epsilon_2 a' 2\bar{\theta}_2) \ ,$$

where we have defined $f' = f|_{M'}$ and $a' = a|_{M'}$. The Lie derivative of $\bar{\theta}_\alpha$ is therefore

$$\mathcal{L}_X \bar{\theta}_\alpha = (-da', -\epsilon_2 a' 2\bar{\theta}_3, \epsilon_2 a' 2\bar{\theta}_2) \ .$$

From (23) it follows that

$$\mathcal{L}_X \omega'_\alpha = \frac{2}{\nu} \left[-\epsilon_\alpha \mathcal{L}_X d\bar{\theta}_\alpha + \epsilon_3 2 \mathcal{L}_X \bar{\theta}_\beta \wedge \bar{\theta}_\gamma + \epsilon_3 2 \bar{\theta}_\beta \wedge \mathcal{L}_X \bar{\theta}_\gamma \right] \ ,$$

which is calculated to be

$$(\mathcal{L}_X \omega'_\alpha)_{\alpha=1,2,3} = (0, -\epsilon_3 2a' \omega'_3, -2a' \omega'_2) \ . \quad (24)$$

The following theorem is proved in [GL, Thm. 2.4] in the quaternionic case and [V, Thm 5.2] in the para-quaternionic case.

Theorem 3. *Let $X \in \mathfrak{X}(M')$ be a Killing vector field on an ε -quaternionic Kähler manifold (M', g', Q) . There exists a unique section $\mu^X \in \Gamma(Q)$ on an open subset $U \subset M'$ such that*

$$\nabla \cdot \mu^X|_U = \omega'_\alpha(X, \cdot) J'_\alpha \ .$$

Proposition 3. *The Lie derivative of ω'_α may be written in terms of $\mu^X =: \sum_{\alpha=1}^3 \mu_\alpha^X J'_\alpha$ as*

$$\mathcal{L}_X \omega'_\alpha = \epsilon_3 (\epsilon_\alpha \nu \mu_\beta^X + \epsilon_\gamma 2 \bar{\theta}_\beta(X)) \omega'_\gamma - \epsilon_3 (\epsilon_\alpha \nu \mu_\gamma^X + \epsilon_\beta 2 \bar{\theta}_\gamma(X)) \omega'_\beta \ . \quad (25)$$

Proof. From (22) it follows that

$$\nabla \cdot \omega'_\alpha = \epsilon_3 2 (\bar{\theta}_\beta(\cdot) \epsilon_\gamma \omega'_\gamma - \bar{\theta}_\gamma(\cdot) \epsilon_\beta \omega'_\beta) \ . \quad (26)$$

Expanding $\mu^X = \sum_{\alpha=1}^3 \mu_\alpha^X J'_\alpha$ and making use of (22) we find

$$\nabla \cdot (\mu_\alpha^X J'_\alpha) = d\mu_\alpha^X J'_\alpha + \epsilon_3 \epsilon_\alpha \mu_\alpha^X 2 (\bar{\theta}_\beta(\cdot) J'_\gamma - \bar{\theta}_\gamma(\cdot) J'_\beta) \ ,$$

and since (J'_1, J'_2, J'_3) are linearly independent it follows from Theorem 3 that

$$d\mu_\alpha^X + \epsilon_3\epsilon_\beta 2\mu_\beta^X \bar{\theta}_\gamma - \epsilon_3\epsilon_\gamma 2\mu_\gamma^X \bar{\theta}_\beta = \iota_X \omega'_\alpha. \quad (27)$$

Equations (26) and (27) together with the fact that $\epsilon_\alpha\epsilon_\beta = -\epsilon_\gamma$ imply

$$\begin{aligned} \text{alt}(\nabla.\omega'_\alpha)(X, \cdot) &= \epsilon_3(\bar{\theta}_\beta \wedge \epsilon_\gamma d\mu_\gamma^X - \bar{\theta}_\gamma \wedge \epsilon_\beta d\mu_\beta^X) \\ &\quad + \epsilon_\alpha 2\mu_\beta^X \bar{\theta}_\beta \wedge \bar{\theta}_\alpha + \epsilon_\alpha 2\mu_\gamma^X \bar{\theta}_\gamma \wedge \bar{\theta}_\alpha, \end{aligned} \quad (28)$$

where alt is the anti-symmetrisation operator, i.e.

$$\text{alt}(\nabla_Y \omega'_\alpha)(X, Z) = \frac{1}{2}((\nabla_Y \omega'_\alpha)(X, Z) - (\nabla_Z \omega'_\alpha)(X, Y)).$$

The Lie derivative of ω'_α may be written as

$$\begin{aligned} \mathcal{L}_X \omega'_\alpha &= \iota_X d\omega'_\alpha + d\iota_X \omega'_\alpha \\ &= \nabla_X \omega'_\alpha - 2 \text{alt}(\nabla.\omega'_\alpha)(X, \cdot) + \epsilon_3 2d(\epsilon_\beta \mu_\beta^X \bar{\theta}_\gamma - \epsilon_\gamma \mu_\gamma^X \bar{\theta}_\beta). \end{aligned}$$

Substituting (26), (27) and (28) into the above expressions produces the desired result. \square

Comparing the two expressions (24) and (25) for the Lie derivative of ω'_α we find

$$\mu_\alpha^X = \left(-\frac{1}{2|f'|}, 0, 0\right) \quad \Rightarrow \quad \mu^X = -\frac{1}{2|f'|} J'_1. \quad (29)$$

The expression (29) is already enough to prove that J'_1 is an integrable ϵ_1 -complex structure. To show this we will adapt the proof of the statement in the case of an almost-complex structure on a quaternionic Kähler manifold given in [B, Prop. 3.3]. Using (27) we have $\bar{\theta}_2 = \epsilon_2 |f'| \iota_X \omega'_3$ and $\bar{\theta}_3 = -\epsilon_2 |f'| \iota_X \omega'_2$, hence

$$\nabla J'_1 = -2|f'|(\iota_X \omega'_2 \otimes J'_2 - \iota_X \omega'_3 \otimes J'_3). \quad (30)$$

Fix a point $x \in M'$. Consider a one-form α that satisfies $(\nabla \alpha)_x = 0$. Define the ϵ_1 -complex one-form $A = \alpha + \epsilon_1 i_{\epsilon_1} J_1'^* \alpha$, where i_{ϵ_1} is the ϵ_1 -complex unit satisfying $i_{\epsilon_1}^2 = \epsilon_1$ and $\bar{i}_{\epsilon_1} = -i_{\epsilon_1}$. The one-form A is J_1' -holomorphic, that is $J_1'^* A = i_{\epsilon_1} A$. Using (30) we have

$$(\nabla A)_x = -\epsilon_1 i_{\epsilon_1} ((\nabla J_1'^*) \alpha)_x = \epsilon_1 i_{\epsilon_1} 2|f'|(\iota_X \omega'_2 \otimes J_2'^* \alpha + \iota_X \omega'_3 \otimes J_3'^* \alpha) \Big|_x,$$

and therefore

$$(dA)_x = \epsilon_1 i_{\epsilon_1} 2|f'|(\iota_X \omega'_2 \wedge J_2'^* \alpha + \iota_X \omega'_3 \wedge J_3'^* \alpha) \Big|_x.$$

We now define

$$\beta_i := 2|f'| \iota_X \omega'_i, \quad \gamma_i^\alpha := J_i'^* \alpha, \quad i = 2, 3,$$

which satisfy

$$J_1'^* \beta_2 = \epsilon_1 \beta_3, \quad J_1'^* \beta_3 = \beta_2, \quad J_1'^* \gamma_2^\alpha = -\gamma_3^\alpha, \quad J_1'^* \gamma_3^\alpha = -\epsilon_1 \gamma_2^\alpha.$$

We may then write

$$(dA)_x = -\frac{\epsilon_1 i_{\epsilon_1}}{2} (B \wedge \bar{C}^\alpha + \bar{B} \wedge C^\alpha)_x,$$

where $B := \beta_2 + \epsilon_1 i_{\epsilon_1} J_1'^* \beta_2$ and $C^\alpha := \gamma_2^\alpha + \epsilon_1 i_{\epsilon_1} J_1'^* \gamma_2^\alpha$. Since B and C^α are J_1' -holomorphic this shows that J_1' is integrable by the Newlander–Nirenberg theorem in the complex case and Frobenius' theorem in the para-complex case. This completes the proof of Theorem 2. \square

2 One-parameter deformations of the local temporal and Euclidean c -maps

In this section we will consider three important examples of the ε -HK/QK correspondence. They are related to constructions in the physics literature known as the local (or supergravity) spatial, temporal and Euclidean c -maps.

Let $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ and $\epsilon_3 = -\epsilon_1 \epsilon_2$ as in Section 1. We start with a conical affine special ϵ_1 -Kähler (CAS ϵ_1 K) manifold M and consider the ε -hyper-Kähler manifold N obtained from the global (or rigid) spatial c -map for $(\epsilon_1, \epsilon_2) = (-1, -1)$, temporal c -map for $(\epsilon_1, \epsilon_2) = (-1, 1)$ or Euclidean c -map for $(\epsilon_1, \epsilon_2) = (1, \pm 1)$. One can identify N with the cotangent bundle of M . Using a natural vector field fulfilling the assumptions of Section 1, we apply the ε -HK/QK correspondence to N and obtain a family of ε -quaternionic Kähler manifolds \bar{N}_c that depends on a real parameter c . For $c = 0$ the result agrees with the ε -quaternionic Kähler manifold obtained from the projective special ϵ_1 -Kähler (PS ϵ_1 K) manifold \bar{M} underlying M via the local spatial, temporal or Euclidean c -map. This construction is summarised in the

following diagram:

$$\begin{array}{ccc}
\begin{array}{c} M \\ \text{CAS}_{\epsilon_1}\text{K}, 2n+2 \end{array} & \xrightarrow{\text{global } c\text{-map}} & \begin{array}{c} N = T^*M \\ \epsilon\text{-HK}, 4n+4 \end{array} \\
\downarrow & & \downarrow \epsilon\text{-HK/QK} \\
\begin{array}{c} \bar{M} \\ \text{PS}_{\epsilon_1}\text{K}, 2n \end{array} & \xrightarrow[\text{(undeformed)}]{\text{local } c\text{-map}} & \begin{array}{c} \bar{N}_{c=0} \in \bar{N}_c \\ \epsilon\text{-QK}, 4n+4 \end{array}
\end{array}$$

In the specific case of the local spatial c -map there is a known one-parameter deformation called the *one-loop deformation* of the local spatial c -map [RSV]. It was shown in [ACDM] that the family of target manifold \bar{N}_c in the above example of the ϵ -HK/QK correspondence with $(\epsilon_1, \epsilon_2) = (-1, -1)$ (i.e. the HK/QK correspondence) and $c \neq 0$ is in exact agreement with the family of target manifolds of the one-loop deformation of the local spatial c -map. On the other hand, no deformations of the local temporal or Euclidean c -maps have appeared in the literature. We will show that the above example of the ϵ -HK/QK with $(\epsilon_1, \epsilon_2) = (-1, 1)$ or $(1, \pm 1)$ and $c \neq 0$ results in two new families of para-quaternionic Kähler manifolds \bar{N}_c that can be understood as one-parameter deformations of the local temporal and Euclidean c -map target manifolds.

Let (M, g_M, J, ∇, ξ) be a conical affine special ϵ_1 -Kähler manifold [CM] of dimension $\dim_{\mathbb{R}} M = 2(n+1)$. We assume that $g(\xi, \xi) > 0$ and that if $\epsilon_1 = -1$ then $g_M|_{\{\xi, J\xi\}^\perp \subset TM} < 0$. Let $X = (X^I) = (X^0, \dots, X^n) : U \subset M \xrightarrow{\sim} \tilde{U} \subset \mathbb{C}^{n+1}$ be a set of conical special ϵ_1 -holomorphic coordinates such that the geometric data on the domain $U \subset M$ is encoded in an ϵ_1 -holomorphic function $F : \tilde{U} \rightarrow \mathbb{C}$ that is homogeneous of degree 2. The metric may then be written as⁷

$$g_M = N_{IJ} dX^I d\bar{X}^J, \quad N_{IJ}(X, \bar{X}) := i_{\epsilon_1}(\bar{F}_{IJ}(\bar{X}) - F_{IJ}(X)) = -2\epsilon_1 \text{Im } F_{IJ}(X), \quad (31)$$

and the Euler vector field as $\xi = X^I \partial_{X^I} + \bar{X}^I \partial_{\bar{X}^I}$, where $F_{IJ}(X) := \frac{\partial^2 F(X)}{\partial X^I \partial X^J}$ for $I, J = 0, \dots, n$. The ϵ_1 -Kähler potential for g_M is given by $r^2 = g_M(\xi, \xi) = X^I N_{IJ}(X, \bar{X}) \bar{X}^J$.

⁷Note that apart from interchanging X and z , we use conventions in this section that agree with [ACDM] for $(\epsilon_1, \epsilon_2) = (-1, -1)$. Compared to [CDMV] N_{IJ} is defined in terms of the ϵ_1 -holomorphic prepotential F with an extra minus sign. The same holds true for the definition of the projective special ϵ_1 -Kähler metric $g_{\bar{M}}$ in terms of the conical affine special ϵ_1 -Kähler metric g_M .

We will assume that ξ and $J\xi$ induce free $\mathbb{R}^{>0}$ - and $A^{J\xi}$ -actions, respectively, where⁸

$$A^{J\xi} := \begin{cases} \{z = x + iy \in \mathbb{C} : |z|^2 = x^2 + y^2 = 1\} \simeq S^1 & \text{if } \epsilon_1 = -1 \\ \{z = x + ey \in C : |z|^2 = x^2 - y^2 = 1\}_0 \simeq \mathbb{R} & \text{if } \epsilon_1 = +1. \end{cases} \quad (32)$$

Notice that $\mathbb{R}^{>0} \times A^{J\xi} \simeq \mathbb{C}^*$ if $\epsilon_1 = -1$, and $\mathbb{R}^{>0} \times A^{J\xi} \simeq C_0^*$ if $\epsilon_1 = 1$. Let

$$\bar{\pi} := M \rightarrow \bar{M} := M/(\mathbb{R}^{>0} \times A^{J\xi}), \quad (33)$$

and define $r := \sqrt{g_M(\xi, \xi)}$. Let $(\bar{M}, -g_{\bar{M}}, J_{\bar{M}})$ be the ϵ_1 -Kähler manifold obtained from the ϵ_1 -Kähler quotient with level set $\{r = 1\} \subset M$. Then $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ is a projective special ϵ_1 -Kähler manifold [CM] that is positive definite if $\epsilon_1 = -1$ and g_M has complex inverse-Lorentz signature. Note that

$$g_M = dr^2 - \epsilon_1 r^2 \tilde{\eta}^2 - r^2 \bar{\pi}^* g_{\bar{M}}, \quad (34)$$

where

$$\tilde{\eta} := \frac{1}{r^2} g_M(J\xi, \cdot) = -\frac{\epsilon_1}{r^2} \omega_1(\xi, \cdot) = d^c \log r = i_{\epsilon_1} (\bar{\partial} - \partial) \log r. \quad (35)$$

Let us assume that $X^0 \bar{X}^0 > 0$ and $\text{Re } X^0 > 0$. Then the $(\mathbb{R}^{>0} \times A^{J\xi})$ -invariant functions $z^\mu := \frac{X^\mu}{X^0}$, $\mu = 1, \dots, n$, define a local ϵ_1 -holomorphic coordinate system on \bar{M} . The ϵ_1 -Kähler potential for $g_{\bar{M}}$ is

$$\mathcal{K} := -\log(z^I N_{IJ}(z, \bar{z}) \bar{z}^J). \quad (36)$$

The spatial $((\epsilon_1, \epsilon_2) = (-1, -1))$, temporal $((\epsilon_1, \epsilon_2) = (-1, 1))$ and Euclidean $((\epsilon_1, \epsilon_2) = (1, \pm 1))$ global c -map assigns an ε -hyper-Kähler manifold to any affine special ϵ_1 -Kähler manifold [CMMS]. We will now review this construction. First of all, note that the real coordinates

$$(q^a)_{a=1, \dots, 2n+2} := (x^I, y_J)_{I, J=0, \dots, n} := (\text{Re } X^I, \text{Re } F_J(X))_{I, J=0, \dots, n} \quad (37)$$

on M are ∇ -affine and fulfil

$$\omega_M := -\epsilon_1 g_M(J\cdot, \cdot) = -2dx^I \wedge dy_I. \quad (38)$$

With respect to the coordinates (q^a) , the function $H := \frac{1}{2} X^I N_{IJ} \bar{X}^J$ on M is a Hesse potential, i.e. $g_M = H_{ab} dq^a dq^b$ where $H_{ab} := \frac{\partial^2 H}{\partial q^a \partial q^b}$. The matrix-valued function $(H_{ab})_{a,b=1, \dots, 2n+2}$

⁸The unit para-complex numbers have four connected components. Here the subscript $_0$ denotes the connected component of $1 \in C$.

and its inverse can be calculated to be

$$(H_{ab}) = \begin{pmatrix} N - \epsilon_1 R N^{-1} R & \epsilon_1 2 R N^{-1} \\ \epsilon_1 2 N^{-1} R & -\epsilon_1 4 N^{-1} \end{pmatrix}, \quad (H^{ab}) = \begin{pmatrix} N^{-1} & \frac{1}{2} N^{-1} R \\ \frac{1}{2} R N^{-1} & \frac{1}{4} (-\epsilon_1 N + R N^{-1} R) \end{pmatrix}, \quad (39)$$

where $R_{IJ} := 2\text{Re } F_{IJ}(X)$, i.e. $F_{IJ} = \frac{1}{2}(R_{IJ} - \epsilon_1 i_{\epsilon_1} N_{IJ})$. We consider the cotangent bundle $N := T^*M$ and introduce real functions $(p_a) := (\tilde{\zeta}_I, \zeta^J)$ on N such that together with the pullback of (q^a) to N , they form a system of canonical coordinates, i.e. $p_a dq^a \in T^*M \mapsto (q^a, p_b)$. The ε -hyper-Kähler structure on N defined in [CMMS] is given by

$$\begin{aligned} g &= H_{ab} dq^a dq^b + \epsilon_1 \epsilon_2 H^{ab} dp_a dp_b \\ \omega_1 &= -\Omega_{ab} dq^a \wedge dq^b + \frac{\epsilon_2}{4} \Omega^{ab} dp_a \wedge dp_b = -2dx^I \wedge dy_I - \frac{\epsilon_2}{2} d\tilde{\zeta}_I \wedge d\zeta^I \\ \omega_2 &= \epsilon_1 2\Omega_{ac} H^{cb} dq^a \wedge dp_b \\ \omega_3 &= dq^a \wedge dp_a = dx^I \wedge d\tilde{\zeta}_I + dy_I \wedge d\zeta^I, \end{aligned} \quad (40)$$

where $(\Omega_{ab}) = -(\Omega^{ab}) = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix}$.

To make contact with the formulation of the local spatial, temporal and Euclidean c -map given in [CDMV] later in this section, we consider the tangent bundle $\tilde{N} := TM$ of M and introduce real functions (\hat{q}^a) on \tilde{N} by $\hat{q}^a \frac{\partial}{\partial q^a} \in TM \mapsto (q^a, \hat{q}^b)$. We identify $N = T^*M$ with $\tilde{N} = TM$ using the ϵ_1 -Kähler form $\omega_M : TM \rightarrow T^*M$, $v \mapsto \omega_M(v, \cdot)$. After this identification, we have

$$\hat{q}^a = \frac{1}{2} \Omega^{ab} p_b. \quad (41)$$

The the ε -hyper-Kähler structure on \tilde{N} is given as follows (note that $\Omega^{ab} H_{bc} \Omega^{cd} = \epsilon_1 4 H^{ad}$):

$$\begin{aligned} g &= H_{ab} dq^a dq^b - \epsilon_2 H_{ab} d\hat{q}^a d\hat{q}^b, \\ \omega_1 &= -\Omega_{ab} dq^a \wedge dq^b - \epsilon_2 \Omega_{ab} d\hat{q}^a \wedge d\hat{q}^b, \\ \omega_2 &= -H_{ab} d\hat{q}^a \wedge dq^b, \\ \omega_3 &= 2\Omega_{ab} dq^a \wedge d\hat{q}^b. \end{aligned} \quad (42)$$

Note that the complex structures are given by

$$\begin{aligned} J_1 &= J^a_b \frac{\partial}{\partial q^a} \otimes dq^b - J^a_b \frac{\partial}{\partial \hat{q}^a} \otimes d\hat{q}^b, \\ J_2 &= \frac{\partial}{\partial \hat{q}^a} \otimes dq^a + \epsilon_2 \frac{\partial}{\partial q^a} \otimes d\hat{q}^a, \\ J_3 &= \epsilon_2 J^a_b \frac{\partial}{\partial q^a} \otimes d\hat{q}^b - J^a_b \frac{\partial}{\partial \hat{q}^a} \otimes dq^b, \end{aligned} \quad (43)$$

where $J^a_b = -\frac{1}{2}\Omega^{ac}H_{cb}$.

Consider the lift of the vector field $-\epsilon_1 2J\xi$ on M to a vector field Z on $\tilde{N} = TM$ such that $Z(\hat{q}^a) = 0$, i.e.

$$Z := -\epsilon_1 H_a \Omega^{ab} \frac{\partial}{\partial q^b} \in \mathfrak{X}(\tilde{N}) . \quad (44)$$

The vector field Z is a J_1 -holomorphic Killing vector field such that $\mathcal{L}_Z J_2 = \epsilon_1 2J_3$. Since

$$d(-\epsilon_1 2H) = -\omega_1(Z, \cdot) ,$$

the function $f = -\epsilon_1(2H - c)$ fulfils $df = -\omega_1(Z, \cdot)$ for any $c \in \mathbb{R}$. We may therefore apply the ε -HK/QK correspondence to \tilde{N} endowed with the above ε -hyper-Kähler structure. We begin by calculating

$$\begin{aligned} \beta = g(Z, \cdot) &= -4q^a \Omega_{ab} dq^b , \quad \beta(Z) = -\epsilon_1 8H , \quad f_1 = f - \frac{1}{2}\beta(Z) = \epsilon_1(2H + c) , \\ \omega_1 - \frac{1}{2}d\beta &= -\epsilon_2 \Omega_{ab} d\hat{q}^a \wedge d\hat{q}^b + \Omega_{ab} dq^a \wedge dq^b . \end{aligned} \quad (45)$$

Let $\pi : P := \mathbb{R} \times \tilde{N} \rightarrow \tilde{N}$ be the trivial \mathbb{R} -bundle over \tilde{N} and let s denote the standard coordinate on the fibre of P such that $X_P = \frac{\partial}{\partial s}$. We define $\tilde{\phi} := -\epsilon_2 2s$. Using the above information we see that the connection one-form

$$\eta := -\epsilon_2 \frac{1}{2} d\tilde{\phi} - \epsilon_2 \hat{q}^a \Omega_{ab} d\hat{q}^b + q^a \Omega_{ab} dq^b \quad (46)$$

has curvature $d\eta = \pi^*(\omega_1 - \frac{1}{2}d\beta)$ and satisfies $\eta(X_P) = 1$. The one-forms defined in Eq. (12) are calculated to be:

$$\begin{aligned} \theta_0^P &= \frac{1}{2}df = -\epsilon_1 dH , \\ \theta_1^P &= \eta + \frac{1}{2}\beta = -\epsilon_2 \frac{1}{2} d\tilde{\phi} - \epsilon_2 \hat{q}^a \Omega_{ab} d\hat{q}^b - q^a \Omega_{ab} dq^b , \\ \theta_2^P &= -\frac{\epsilon_2}{2} \omega_3(Z, \cdot) = \epsilon_1 \epsilon_2 H_a d\hat{q}^a , \\ \theta_3^P &= \frac{\epsilon_2}{2} \omega_2(Z, \cdot) = -\epsilon_2 2q^a \Omega_{ab} d\hat{q}^b . \end{aligned} \quad (47)$$

The metric $g_P = \frac{2}{f_1} \eta^2 + g$ is given by

$$g_P = H_{ab} (dq^a dq^b - \epsilon_2 d\hat{q}^a d\hat{q}^b) + \epsilon_1 \frac{2}{(2H + c)} \left(\frac{1}{2} d\tilde{\phi} + \hat{q}^a \Omega_{ab} d\hat{q}^b - \epsilon_2 q^a \Omega_{ab} dq^b \right)^2 . \quad (48)$$

A degenerate tensor field \tilde{g} on P that restricts to the ε -quaternionic Kähler metric g' given in Eq. (13) on any appropriate submanifold M' is given by

$$\begin{aligned} \epsilon_1 2\sigma \tilde{g} = & \tilde{H}_{ab} (dq^a dq^b - \epsilon_2 d\hat{q}^a d\hat{q}^b) \\ & + \epsilon_1 \epsilon_2 \frac{8}{(2H - c)^2} (q^a \Omega_{ab} d\hat{q}^b)^2 - \epsilon_1 \frac{4}{2H(2H - c)} (q^a \Omega_{ab} dq^b)^2 \\ & - \epsilon_1 \frac{8H}{(2H - c)^2(2H + c)} \left[\left(\frac{1}{2} d\tilde{\phi} + \hat{q}^a \Omega_{ab} d\hat{q}^b \right) + \epsilon_2 \frac{c}{2H} q^a \Omega_{ab} dq^b \right]^2, \end{aligned} \quad (49)$$

where $\tilde{H}_{ab} := \frac{\partial^2}{\partial q^a \partial q^b} \tilde{H}$, and $\tilde{H} := -\frac{1}{2} \log(2H - c)$. The horizontal lift of $Z \in \mathfrak{X}(M)$ to $\tilde{Z} \in \mathfrak{X}(P)$ is given by

$$\tilde{Z} = Z - \eta(Z) X_P = -\epsilon_1 H_a \Omega^{ab} \frac{\partial}{\partial q^b} - \epsilon_1 2H X_P.$$

The fundamental vector field is given by $X_P = -\epsilon_2 2 \frac{\partial}{\partial \tilde{\phi}}$, and therefore the vector field $Z_1^P \in \mathfrak{X}(P)$ is given by

$$Z_1^P = \tilde{Z} + f_1 X_P = -\epsilon_1 H_a \Omega^{ab} \frac{\partial}{\partial q^b} - \epsilon_1 \epsilon_2 2c \frac{\partial}{\partial \tilde{\phi}}. \quad (50)$$

As a corollary of Theorem 1, we obtain that the restriction of the tensor field given in Eq. (49) to any codimension one submanifold $M' \subset \tilde{N}$ that is transversal to the vector field given in Eq. (50) defines an ε -quaternionic Kähler metric for any $c \in \mathbb{R}$ (after restriction to open subsets where $2H - c$ and $2H + c$ have constant sign).

Setting $c = 0$ in Eq. (49) reproduces the formula for the target metrics of the local spatial, temporal and Euclidean c -maps in [CDMV, Sec. 4.2] up to an overall factor given by⁹ $\tilde{g} = \frac{\epsilon_1 \sigma}{2} g'^{[CDMV]}$. This shows that the families of ε -quaternionic Kähler manifolds defined above describe one-parameter deformations of the local spatial, temporal and Euclidean c -map metrics.

2.1 Ferrara-Sabharwal form

In this subsection we will write Eq. (49) in an alternative system of coordinates. This will, in particular, make manifest that for the case $(\epsilon_1, \epsilon_2) = (-1, -1)$ the quaternionic Kähler

⁹This implies that the reduced scalar curvatures are related by $\nu = \epsilon_1 2\sigma \nu^{[CDMV]}$. From Remark 1 we have $\nu = -\epsilon_1 4\sigma$, which is consistent with $\nu^{[CDMV]} = -2$.

metric obtained from Eq. (49) agrees with the one-loop deformation [RSV] of the original Ferrara-Sabharwal c -map metric [FS].

We use the following system of coordinates on the conical affine special ϵ_1 -Kähler manifold M :

$$\left(r = \sqrt{X^I N_{IJ} \bar{X}^J}, \phi := \arg X^0 = -\frac{\epsilon_1 i_{\epsilon_1}}{2} (\log \bar{X}^0 - \log X^0), z^\mu = \frac{X^\mu}{X^0} \right)_{\mu=1, \dots, n}. \quad (51)$$

The inverse coordinate transformation is given by $X^I = \frac{r e^{i_{\epsilon_1} \phi}}{\sqrt{z^I N_{IJ} \bar{z}^J}} z^I = r e^{\mathcal{K}/2} e^{i_{\epsilon_1} \phi} z^I$, where $z^0 := 1$. In these coordinates, we have $J\xi = \partial_\phi$ and

$$\tilde{\eta} = \frac{1}{r^2} g_M(J\xi, \cdot) = -\epsilon_1 d\phi - \frac{1}{2} d^c \mathcal{K}, \quad (52)$$

where $d^c = i_{\epsilon_1}(\bar{\partial} - \partial)$.

Now, we translate each term in Eq. (49) into the set of coordinates $(\rho, \tilde{\varphi}, z^\mu, \tilde{\zeta}_I, \zeta^J)$ used in [ACDM, Eq. (4.11)] (generalised to the case where z^μ may be para-holomorphic). Let us define

$$\rho := r^2 - c = 2H - c, \quad \tilde{\varphi} := -2\tilde{\phi}, \quad (p_a) = (2\Omega_{ab} \hat{q}^b) = (\tilde{\zeta}_I, \zeta^J). \quad (53)$$

In these coordinates the vector field Z_1^P in Eq. (50) is given by $Z_1^P = -\epsilon_1 2 \frac{\partial}{\partial \phi} - \epsilon_1 \epsilon_2 2c \frac{\partial}{\partial \tilde{\phi}}$. Hence, we can choose $M' := \{\phi = 0\} \subset P$ as a codimension one submanifold transversal to Z_1^P .

Using the fact that $\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I = -4\hat{q}^a \Omega_{ab} d\hat{q}^b$ and

$$d^c \mathcal{K} = -2\tilde{\eta}|_{M'} = \frac{2\epsilon_1}{r^2} \omega_M(\xi, \cdot)|_{M'} = -\frac{2\epsilon_1}{H} q^a \Omega_{ab} dq^b|_{M'}, \quad (54)$$

the last term in Eq. (49) is given by $-\epsilon_1 \frac{1}{4\rho^2} \frac{\rho+c}{\rho+2c} (d\tilde{\varphi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I + \epsilon_1 \epsilon_2 c d^c \mathcal{K})^2$ after restricting to M' . Next, we calculate (using $dr^2 = \frac{1}{4(\rho+c)} d\rho^2$, $dH = \frac{1}{2} d\rho$ and Eqs. (35), (54))

$$\begin{aligned} & \tilde{H}_{ab} dq^a dq^b - \frac{2\epsilon_1}{H(2H-c)} (q^a \Omega_{ab} dq^b)^2 \Big|_{M'} \\ &= -\frac{1}{2H-c} g_M + \frac{2(dH)^2}{(2H-c)^2} - \frac{2\epsilon_1}{H(2H-c)} (q^a \Omega_{ab} dq^b)^2 \Big|_{M'} \\ &= \frac{\rho+2c}{4\rho^2(\rho+c)} d\rho^2 + \frac{\rho+c}{\rho} g_{\bar{M}}. \end{aligned} \quad (55)$$

To translate the remaining terms, let us define one-forms $A_I := d\tilde{\zeta}_I + F_{IJ}d\zeta^J$ and note that $H_{ab}d\hat{q}^a d\hat{q}^b = -\epsilon_1 H^{ab} dp_a dp_b = -\epsilon_1 A_I N^{IJ} \bar{A}_J$. Also note that, similarly to [ACDM, Lemma 3], one can prove that

$$-A_I N^{IJ} \bar{A}_J + \frac{2}{\rho + c} (X^I A_I) (\bar{X}^I \bar{A}_I) = -\frac{\epsilon_1}{2} \hat{H}^{ab} dp_a dp_b, \quad (56)$$

where $(\hat{H}^{ab}) = \begin{pmatrix} \mathcal{J}^{-1} & \mathcal{J}^{-1} \mathcal{R} \\ \mathcal{R} \mathcal{J}^{-1} & -\epsilon_1 \mathcal{J} + \mathcal{R} \mathcal{J}^{-1} \mathcal{R} \end{pmatrix}$ is defined in terms of the real matrix-valued functions $\mathcal{R} = (\mathcal{R}_{IJ})$, $\mathcal{J} = (\mathcal{J}_{IJ})$ defined by

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i_{\epsilon_1} \mathcal{J}_{IJ} := \bar{F}_{IJ} - \epsilon_1 i_{\epsilon_1} \frac{N_{IK} X^K X^L N_{LJ}}{X^K N_{KL} X^L}. \quad (57)$$

Note that \mathcal{J} and \mathcal{R} are well-defined both on M and on \bar{M} . Using this information we calculate

$$\begin{aligned} & -\epsilon_2 \tilde{H}_{ab} d\hat{q}^a d\hat{q}^b + \epsilon_1 \epsilon_2 \frac{8}{(2H - c)^2} (q^a \Omega_{ab} d\hat{q}^b)^2 \\ &= \frac{\epsilon_2}{2H - c} H_{ab} d\hat{q}^a d\hat{q}^b - \frac{2\epsilon_2}{(2H - c)^2} (H_a d\hat{q}^a)^2 + \frac{8\epsilon_1 \epsilon_2}{(2H - c)^2} (q^a \Omega_{ab} d\hat{q}^b)^2 \\ &= -\frac{\epsilon_1 \epsilon_2}{\rho} A_I(X) N^{IJ} \bar{A}_J(\bar{X}) - \frac{2\epsilon_2}{\rho^2} (\text{Im } X^I A_I)^2 + \frac{2\epsilon_1 \epsilon_2}{\rho^2} (\text{Re } X^I A_I)^2 \\ &= -\frac{\epsilon_1 \epsilon_2}{\rho} A_I(X) N^{IJ} \bar{A}_J(\bar{X}) + \frac{2\epsilon_1 \epsilon_2}{\rho^2} (X^I A_I) (\bar{X}^I \bar{A}_I) \\ &= -\frac{\epsilon_2}{2\rho} \hat{H}^{ab} dp_a dp_b + \frac{2\epsilon_1 \epsilon_2 c}{\rho^2 (\rho + c)} |X^I A_I(X)|^2 \\ &= -\frac{\epsilon_2}{2\rho} \hat{H}^{ab} dp_a dp_b + \frac{2\epsilon_1 \epsilon_2 c}{\rho^2} e^{\mathcal{K}} |z^I A_I(z)|^2. \end{aligned} \quad (58)$$

Putting everything together, we find that the expression given in Eq. (49) restricts to the following metric on M' :

$$\begin{aligned} g_{\varepsilon FS}^c &= \frac{\rho + c}{\rho} g_{\bar{M}} + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 - \epsilon_1 \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} (d\tilde{\varphi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + \epsilon_1 \epsilon_2 c d^c \mathcal{K})^2 \\ &\quad - \frac{\epsilon_2}{2\rho} \sum dp_a \hat{H}^{ab} dp_b + \frac{2\epsilon_1 \epsilon_2 c}{\rho^2} e^{\mathcal{K}} \left| \sum (z^I d\tilde{\zeta}_I + F_I(z) d\zeta^I) \right|^2, \end{aligned} \quad (59)$$

which is defined on the two domains $\{\rho > \max\{0, -2c\}\}$ and $\{-c < \rho < \max\{0, -2c\}\}$ in $\bar{M} \times \mathbb{R}^{2n+4}$, where $(\rho, \tilde{\varphi}, \tilde{\zeta}_I, \zeta^J)$ are standard coordinates on the second factor. For $(\epsilon_1, \epsilon_2) = (-1, -1)$ this agrees with [ACDM, Eq. (4.11)] and, hence, with the one-loop deformed local c -map metric derived in [RSV].

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